

A COMPUTER SIMULATION OF THE PARADOX  
OF VOTING

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# A COMPUTER SIMULATION OF THE PARADOX OF VOTING\*

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This paper presents estimates of the probability that the occurrence of the Paradox of Voting,<sup>1</sup> commonly known as Arrow's Paradox, will prevent the selection of a majority issue when odd-sized committees of  $m$  judges vote upon  $n$  issues. The estimates, obtained through computer simulation of the voting process, indicate that the probability of such an intransitive social ordering is lower than the ratio of intransitive outcomes to all outcomes.

Many of the arguments in political theory and welfare economics dealing with the paradox (e.g., Downs, 1957; Black, 1958; Schubert, 1960) seem to have implicitly assumed that since the paradox exists, its likelihood of occurrence is very close to 1. The results in this paper may call for a re-examination of these positions.

## THE PARADOX OF VOTING

Consider a panel of three judges—I, II, and III—voting upon three issues— $A$ ,  $B$ , and  $C$ . Each judge ranks the issues in order of preference. Throughout this discussion we will assume that the following conditions hold:

Condition I. For each *individual* judge, the preference relation is *transitive*.<sup>2</sup> That is, if a judge prefers  $A$  to  $B$  and  $B$  to  $C$ , then he prefers  $A$  to  $C$ .

Condition II. For each *individual* judge, the issues are *strongly ordered*: indifference between issues is not allowed.

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<sup>1</sup> The paradox was discovered in the 18th Century by Condorcet. A history of its rediscovery and analysis is contained in Black (1958). An extensive review and interpretation of recent developments is presented by Riker (1961).

<sup>2</sup> By requiring the judges to rank order issues we ensure transitivity for individuals. However, there is much evidence (May, 1954; Rose, 1957; Edwards, 1953; Morrison, 1962; Quandt, 1956)

We will call the result of the voting by the judges a *social ordering* of the issues  $A$ ,  $B$ , and  $C$ . The social ordering is determined by simple majority voting between pairs of issues; i.e., if  $X$  is preferred to  $Y$  by a majority of the three judges, then  $X$  is preferred to  $Y$  in the social ordering.

Figure 1 illustrates a possible voting situation. Each row indicates the ordering of the issues by a judge. Thus, judge II prefers  $B$  to  $A$ ,  $A$  to  $C$ , and  $B$  to  $C$ . In this *voting pattern*, as we shall call it,  $A$  is preferred to  $B$  by a majority of the judges,  $B$  is preferred to  $C$  by a majority of the judges, and  $A$  is preferred to  $C$  by a majority of the judges. Consequently, the resulting social ordering of the issues,  $A$ ,  $B$ , and  $C$  is transitive.

I	A	B	C
II	B	A	C
III	A	C	B

Fig. 1. A voting pattern yielding a transitive social ordering.

I	A	B	C
II	C	A	B
III	B	C	A

Fig. 2. A voting pattern yielding an intransitive social ordering.

Now consider Figure 2 in this pattern,  $A$  is preferred to  $B$ , and  $B$  is preferred to  $C$  by a majority of the judges, but  $C$  is preferred to  $A$  by a majority. The resulting social ordering is intransitive. As Riker (1961) says, "This is the paradox: that the summation of transitive preferences sometimes produces a circular [intransitive] result." (p. 901)

This paradox may occur with  $m$  judges and  $n$  issues whenever  $m$  and  $n$  are both greater than 2. We will add a third condition so that the

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that if judges make pairwise choices among issues, individual intransitivity often results. We can account for individual intransitivity in part by constructing a hypothetical "internal voting process" for each individual. Here the "judges" would be various criteria along which issues could be ranked. Even if ordering along a single criterion is transitive, the output that we observe may yield intransitive individual preferences.

strong ordering applies to the social ordering as well as to individual orderings.

Condition III. All issues in the social ordering are strongly ordered.

To ensure this condition, we will consider only voting patterns where the number of judges,  $m$ , is odd. This precludes a voting pattern where the social ordering is indifferent between  $X$  and  $Y$ ; i.e., where  $X$  is preferred to  $Y$  as often as  $Y$  is preferred to  $X$ .

If  $n$ , the number of issues, is greater than 3, then we can classify the intransitive social orderings that may occur into two categories:

*Type 1.* In these social orderings there is an intransitivity between at least 3 issues, but there is still an issue that is preferred to all others by a majority of the judges. Figure 3 shows a voting pattern that yields such a social ordering.

I	A	B	C	D
II	A	D	B	C
III	C	A	D	B

FIG. 3. Voting pattern for 3 judges and 4 issues yielding a Type 1 intransitive social ordering.

$A$  is preferred to all other issues by a majority of the judges, although the social ordering among  $B$ ,  $C$ , and  $D$  is intransitive.

*Type 2.* In this type of intransitive social ordering there is no issue that is preferred to all others by a majority of the judges. Figure 4 illustrates a voting pattern that yields a Type 2 intransitive social ordering:  $B$  is preferred to  $C$  by two judges;  $C$  is preferred to  $A$  by two judges, but  $A$  is preferred to  $B$  by two judges;  $A$ ,  $B$ , and  $C$  are all preferred to  $D$  by two out of three judges. Figure 2 also illustrates a Type 2 intransitivity.<sup>3</sup> (The rationale for this particular classification will be developed below. For other purposes one might wish to categorize intransitive patterns differently. For instance,

<sup>3</sup> In set-theoretic terms:

- $U$  ≡ universe of all social orderings
- $A$  ≡ set of all transitive social orderings
- $B$  ≡ set of all intransitive social orderings
- $b_1$  ≡ set of all Type 1 intransitive social orderings
- $b_2$  ≡ set of all Type 2 intransitive social orderings

The following relations hold:

$$A \cup B = U$$

$$A \cap B = \phi$$

$$b_1 \cup b_2 = B$$

$$b_1 \cap b_2 = \phi$$

for  $n = 3$ :  $b_1 = \phi$ ,  $b_2 = B$   
 $\phi$  = empty set.

in Figure 4 all issues are preferred to  $D$  by a majority of the judges, thus  $D$  is the "majority loser," even though there is no majority winner.)

I	A	B	D	C
II	C	A	D	B
III	B	C	A	D

FIG. 4. Voting pattern for 3 judges and 4 issues yielding a Type 2 intransitive social ordering.

The existence of the paradox has concerned political scientists and economists for many years. There are two general views of the consequences that follow from the fact that the summation of transitive individual preferences may result in intransitive social preferences. Economists, searching for a mechanism whereby a social welfare function can be derived from individual utility functions, are concerned with occurrence of *any* intransitivity in the social ordering. Thus, they are usually interested in the frequency of occurrence of either Type 1 or Type 2 intransitive social orderings.

Political scientists, especially those dealing with the theory of voting, are often concerned with a voting procedure that will yield a single most preferred issue. Riker (1961) says, "but for the theory of voting one wishes to know not the amount of possible inconsistency, but the frequency with which inconsistency prevents majority decision," i.e., the frequency of Type 2 intransitive social ordering.

This research is directed toward the discovery of a function  $P(m, n)$  that expresses the probability of obtaining a Type 2 intransitivity, when  $m$  judges vote upon  $n$  issues under conditions I through III. We will analyze voting patterns in which we assume that all of the  $n!$  possible individual orderings are equally likely to occur. This assumption has two equivalent interpretations: 1) All judges are equally likely to have any one of the  $n!$  possible individual preference orderings among  $n$  issues; 2) We obtain the  $m$  judges by sampling randomly with replacement from a population of judges uniformly distributed among the  $n!$  possible individual orderings of  $n$  issues.

The first interpretation implies that judges are in fact indifferent among the  $n$  alternatives and that when condition II requires a ranking, all judges are equally likely to arrive at any of the  $n!$  individual orderings. The second interpretation does not require any individual to be indifferent among the alternatives but requires only that there be a uniform distribution of potential judges over all possible preference orderings. The latter interpretation, while

somewhat more realistic, does not escape the criticism that the equally likely conditions rarely obtain in actual voting situations. Furthermore, as recently demonstrated by Riker (1965), when the paradox is consciously generated by the judges, then *a priori* calculations based upon any kind of assumed probability distribution are inappropriate.<sup>4</sup> However, the equally likely assumption provides a great analytic simplification and so will be used throughout this paper.

It is important to note that this is an assumption that all possible *individual* orderings are equally likely. This is not equivalent to, nor does it imply that, all *social* orderings (as defined above) are equally likely. One of the main points of this paper is the demonstration that equally likely individual orderings do not aggregate to equally likely social orderings.

Several investigators have proposed expressions that give the proportion of intransitive social orderings to all possible social orderings. We will call this proportion  $P(n)$ , since it is a function of the number of issues under consideration.

Duncan Black (1958) reviews Condorcet's investigation of the proportion of all intransitivities (Type 1 or Type 2). For  $n=3$  there are only Type 2 intransitivities and Condorcet's results are directly relevant to our discussion. The argument runs as follows:

There are

$$K = \binom{n}{2}$$

possible pairwise relations of  $n$  issues, i.e., for  $n=3$ , the

$$K = \binom{3}{2} = \frac{3!}{2!1!} = 3$$

relations are  $ArB$  ( $A$  has some relationship to  $B$ ),  $ArC$ , and  $BrC$ . Each relationship is two-valued; i.e.,  $ArB$  could mean either " $A$  is preferred to  $B$ " or " $B$  is preferred to  $A$ ." A social ordering consists of a selection of one value from each of the  $K$  two-valued relationships. Thus, there are  $s$  possible social orderings, where

$$s = 2^K = 2^{\binom{n}{2}} = \frac{n!}{2^{(n-2)!2!}} = 2^{\frac{n(n-1)}{2}}$$

The number of social orderings that are transitive for all  $n$  issues is  $n!$ . The number of intransitive

social orderings is  $s - n!$ , and the proportion of intransitive orderings is

$$P(n) = \frac{s - n!}{s} = \frac{2^{(n^2-n)/2} - n!}{2^{(n^2-n)/2}}$$

For

$$n=3, P(3) = (8-6)/8 = \frac{1}{4}.$$

Riker (1961), in discussing the frequency with which intransitive social orderings prevent majority decisions (i.e., Type 2 intransitivities), suggests a series for  $P(n)$  based upon Condorcet's argument. Some of Riker's results for  $P(n)$  are presented with ours in Figure 6. Referring to his table, Riker says:

The main inference to be drawn from this table . . . is that the *a priori* expectation of the existence of a majority decision, if it could be calculated, would be quite small, at least for  $n \geq 5$  (p. 908.)

$P(n)$  is the *ratio* of intransitive social orderings to all social orderings. However, it is equivalent to the relative frequency or *a priori* expectation of intransitivities only if we assume that all social orderings are equally likely. But this is a severe constraint to place upon the outcome of the very process that is under investigation. The significance of the paradox rests upon the relationship between the distribution of individual orderings and the distribution of the social orderings that result from their aggregation. We can demonstrate that if individual orderings are equally likely, then the social orderings generated by the pairwise majority voting process are not equally likely. In particular, the intransitive social orderings are less likely than the transitive orderings; hence the probability of an intransitive social ordering,  $P(m, n)$ , is less than the ratio of intransitive orderings to all orderings,  $P(n)$ , for all  $n$ .

Riker and Condorcet have shown that for  $n=3$ ,  $P(3) = \frac{1}{4}$ . We will analyze the voting patterns for 3 judges and 3 issues and show that  $P(m, n) = P(3, 3) = 1/18$ . Black discovered this in his unsuccessful attempt to find an expression for  $P(m, n)$ , but does not indicate his method. He assumes that  $P(m, n)$  will increase rapidly with  $n$ :

It would be of interest to know in what fraction of all the possible cases one motion [issue] is able to get a simple majority against each of the others as  $m$  and  $n$  vary. . . . If the general series could be discovered it would almost certainly show that for a committee with a given number of members [judges], the proportion of cases in which there is no majority decision increases rapidly with an increase in the number of motions [issues] (p. 51.)

<sup>4</sup> I am indebted to Professor Riker for calling my attention to his recent paper and for indicating its implication for these assumptions.

A judge may order the  $n$  issues in  $n!$  different ways. A vote consists of each of  $m$  judges selecting one of the  $n!$  orderings. Thus there are  $(n!)^m$  different voting patterns. For  $n=m=3$  there are 216 equally likely patterns. From these patterns we obtain, through the simple majority voting process, the  $s=2^k=2^3=8$  distinct social orderings. Although 2 of these 8 orderings yield intransitivities, the different social orderings are not equiprobable. Out of 216 patterns, only 12 give rise to the two intransitive social orderings. We can demonstrate this as follows:

The only two general forms of 3 by 3 voting patterns that yield an intransitivity are shown in Figure 5.

$P$	$Q$	$R$	$P$	$Q$	$R$
$R$	$P$	$Q$	$Q$	$R$	$P$
$Q$	$R$	$P$	$R$	$P$	$Q$

FIG. 5. General form of the two voting patterns yielding intransitivity for 3 judges and 3 issues.

In each general pattern, the issues  $A$ ,  $B$ , and  $C$  can be assigned to the positions  $P$ ,  $Q$ ,  $R$  in  $3!$  different ways. Thus there are  $2 \times 3! = 12$  patterns that yield an intransitive social ordering.  $P(3, 3)$  is  $12/216 = 1/18 = .0555 \dots$ . A table of the eight social orderings and their relative frequencies for the 3 by 3 case is presented in Appendix II.

#### COMPUTER SIMULATION

The problems encountered in seeking an analytic expression for  $P(m, n)$  are formidable. Such an expression has as yet not been found.<sup>5</sup> However, we can investigate some of the properties of  $P(m, n)$  by using a digital computer to simulate the voting process. The computer can provide us with empirical estimates of the frequency of occurrence of various types of voting patterns. A program has been written that generates voting patterns and determines whether or not they yield a majority issue in the social ordering. (Appendix III contains a description of the program.) The most straightforward approach is simply to generate and evaluate each of the  $(n!)^m$  distinct patterns for various values of  $m$  and  $n$ . However, the number of patterns increases rapidly with  $m$  and  $n$ : for  $m=n=3$  there are 216 patterns; for  $m=5$ ,  $n=4$  there are over 7 million patterns. If the computer could generate and evaluate a thou-

<sup>5</sup> For an interesting analysis of some features of  $P(m, n)$  see Nicholson (1965). He presents necessary and sufficient conditions for intransitive social orderings in the case of  $m$  judges and 3 issues.

sand patterns per second, it would still take over two hours for the 5 by 4 case.

Rather than enumerate all cases, we have written a program that generates and evaluates a random sample from the complete population of  $(n!)^m$  patterns. With a sufficiently large sample size, we can obtain good estimates of the actual values of  $P(m, n)$ . From a simple statistical analysis (see Appendix I) we have chosen to generate 10,000 patterns for each estimate of  $P(m, n)$  for  $3 \leq n \leq 6$  and odd  $m$ ,  $3 \leq m \leq 7$ . In its present form, the program generates an  $n!$  by  $n$  table of permutations and then samples randomly from this table to obtain votes for each judge in every pattern. Sampling from a table of permutations takes less time than generating the permutations at random for each individual vote. Because a table of  $7!$  by 7 elements would exceed the memory capacity of the computer, for  $n > 6$  we would be forced to trade time for size and generate the permutations at random. For larger values of  $m$ , the running time exceeds 15 minutes per 10,000 votes, and we have chosen to discontinue the runs, since it has become evident that this program is very costly for large values of  $m$ .<sup>6</sup>

The enumerative program was run for  $P(3, 3)$  and  $P(3, 4)$ . For the remaining cases in Figure 6, the sampling program was run, generating 10,000 patterns for each pair of values for  $m$  and  $n$ .

#### RESULTS

Figure 6 shows estimates of  $P(m, n)$ : the probability of occurrence of an intransitive social ordering that prevents majority decision when each of the  $n!$  individual preference orderings is equally likely to be chosen by each of the  $m$  judges. Riker's results for  $P(n)$ , the ratio of Type 2 outcome to all outcomes, are shown in the last row for purposes of comparison. The first two entries in the first row ( $m=3$ ,  $n=3, 4$ ) are exact, having been computed by the enumerative program.<sup>7</sup> The remaining entries are

<sup>6</sup> A somewhat different trade-off between time and space has been taken, at the cost of accuracy, by Campbell and Tullock (1965). In an approach quite similar to ours, they have written a program that gives estimates of  $P(m, n)$  for  $m$  up to 29 and  $n$  up to 17. Because they sample only 1,000 patterns for most of their points, their estimates are somewhat less accurate than ours; however, their results indicate the general shape of  $P(m, n)$  for a wider range of  $m$  and  $n$  than we have considered.

<sup>7</sup> For  $P(3, 3)$ , 12 out of 216 cases are Type 2 intransitivities; for  $P(3, 4)$ , 1,536 out of 13,824

		Number of Issues ( <i>n</i> )				
		3	4	5	6	
Number of Judges ( <i>m</i> )	3	.0555 . . .	.111 . . .	.16	.20	<i>P</i> ( <i>m</i> , <i>n</i> )
	5	.07	.14	.20	.25	
	7	.08	.15	.22	.27	
			.25	.50	.69	.81

FIG. 6. *P*(*m*, *n*): probability of obtaining an intransitive social ordering that prevents majority issue from appearing, for *m* judges and *n* issues. *P*(*n*): ratio of intransitive social orderings that prevent majority issue from appearing. (From Riker, 1961, Column 4, Table I, p. 909.)

estimates; there is at least a .95 probability that they are within .01 of the true value of *P*(*m*, *n*). (See Appendix I.)

DISCUSSION

The primary motivation for this study was the attempt (still unsuccessful) to find an analytic expression for *P*(*m*, *n*). The results presented here provide a first step in exhibiting the behavior of the function for small values of *m* and *n*. In principle, the enumerative program could find the exact number of occurrences of intransitivities for all *m* and *n*, or the sampling program could compute estimates of *P*(*m*, *n*) to any desired degree of accuracy. The only limitations are computer time and space. Perhaps the most useful theoretical result comes from the observation that for any *n*, the *s* different intransitive social orderings are not equiprobable, and depend upon *m*. The correct analytic expression for *P*(*m*, *n*) will have to reflect this fact.

From a more pragmatic point of view, the results demonstrate that under those circumstances where the equiprobable assumption is tenable, attempts at meaningful summations of individual preferences are not doomed to almost certain failure, but instead may often (e.g., at least four times out of five, for 3 judges and 6 issues) be successful. For example, we can apply these results to the problem of determining committee sizes and agenda. A committee of 3 judges is three times as likely to reach an impasse when they consider 5 issues as when they consider 3 issues. However, increasing a committee size from 3 members to 5 members has a relatively small effect on the probability of an intransitive social ordering.

cases are Type 2 intransitivities. The sampling program produced *P*(3, 3) = .056, *P*(3, 4) = .116. These results are within the estimated sampling error.

For those cases where some other distribution of individual orderings is more appropriate, the simulation program can be modified to use the assumed distribution to generate voting patterns. In this manner, we can obtain estimates of the distribution of social orderings for any given distribution of individual orderings.

APPENDIX I

DETERMINATION OF SAMPLE SIZE FOR SAMPLING PROGRAM

The random generation of voting patterns is, in effect, sampling with replacement from a binomially-distributed population. Each item in the population is a voting pattern. For each pair of values for *m* and *n* we estimate *P*, the fraction of items in the population that yield Type 2 intransitivities.

In a random sample of *N* patterns from a population that has an actual proportion *P* of Type 2 intransitive social orderings, the estimate of *P*, *P*\*, is approximately normally distributed with mean *P* and variance  $\sigma^2 = P(1-P)/N$ . We desire a .05 confidence level that *P*\* is within .01 of *P*. That is, we want  $\text{Prob}(|P^* - P| \leq .01) \geq .95$ . Thus we require that  $2\sigma = .01$

or

$$\sigma = .005$$

Then

$$\sigma^2 = 25 \cdot 10^{-6} = P(1 - P)/N.$$

Solving for *N*,

$$N = P(1 - P) \cdot 4 \cdot 10^4.$$

Since *P* is unknown, we consider the worst case, when *P*(1-*P*) is a maximum,  $0 \leq P \leq 1$ . This occurs at *P* = .5; thus, the maximum value of *N* required to obtain the desired confidence level is

$$N = (1/4) \cdot 4 \cdot 10^4 = 10,000.$$

This is the sample size used throughout the runs for the sampling program.

APPENDIX II  
EVALUATION OF 3 BY 3  
VOTING PATTERNS

From an enumeration of all 216 patterns with 3 judges and 3 issues, we get the following results:

Social Ordering*			Intransitive?	Number of Occurrences
<i>ApB</i>	<i>ApC</i>	<i>BpC</i>	no	34
<i>ApB</i>	<i>ApC</i>	<i>CpB</i>	no	34
<i>ApB</i>	<i>CpA</i>	<i>BpC</i>	yes	6
<i>ApB</i>	<i>CpA</i>	<i>CpB</i>	no	34
<i>BpA</i>	<i>ApC</i>	<i>BpC</i>	no	34
<i>BpA</i>	<i>ApC</i>	<i>CpB</i>	yes	6
<i>BpA</i>	<i>CpA</i>	<i>BpC</i>	no	34
<i>BpA</i>	<i>CpA</i>	<i>CpB</i>	no	34
				216

Probability of Intransitive Social Ordering =  $12/216 = .055555 \dots$

\*  $YpX = Y$  is preferred to  $X$  by two or more judges.

APPENDIX III  
PROGRAM DESCRIPTION

The program was written in ALGOL-20, a dialect of ALGOL-60 (Naur, 1963), and run on the CDC-21 computer at the Carnegie Institute of Technology Computation Center. This is a description of the sampling program. The enumerative program is quite similar, and the evaluation mechanism is identical. Both programs are available upon request.

For a given pair of values for  $m$  and  $n$ , ( $m, n \geq 3$ ;  $m$  odd) the program generates and evaluates 10,000 voting patterns. A count is maintained of the number of transitive and intransitive social orderings; the final output is the relative frequency of intransitive social orderings (Type 2). Additional output, in the form of typical voting patterns and their evaluation is provided at every 500th pattern.

For each pair of values  $m, n$  the program executes three main phases: initialization, generation and evaluation. During the initialization phase, a permutation array,  $H$ , is filled with all permutations of the first  $n$  integers. The  $H$  array represents all the possible orderings that a judge might put upon the  $n$  issues.  $H$  has  $n!$  rows and  $n$  columns. For  $n = 3$ , the permutation array looks as follows:

Column	1	2	3
Row 1	1	2	3
Row 2	1	3	2
Row 3	2	1	3
Row 4	2	3	1
Row 5	3	1	2
Row 6	3	2	1

Once the initialization phase is complete, the program generates and evaluates 10,000 voting patterns.

A vote is represented by  $H$  and by  $V$ , the vote vector.  $V$  contains  $m$  elements. Each element has an integer value between 1 and  $n!$ . A vote is generated by assigning a random integer,  $1 \leq J \leq n!$ , to each of the  $m$  elements in  $V$ . When  $V_i = J$ , the vote of judge  $i$  can be obtained from row  $J$  in  $H$ .

For example, let  $m = 5, n = 3$ . If, after the generation of a vote,  $V_1 = 2, V_2 = 6, V_3 = 3, V_4 = 2, V_5 = 1$ , then the voting pattern is:

I	1	3	2
II	3	2	1
III	2	1	3
IV	2	3	1
V	1	2	3

Patterns are evaluated by a scan that searches for an issue that is not preceded by any other issue in more than half of the rows. If there is such an issue, then it is the majority issue and the voting pattern does not yield a Type 2 intransitivity. If there is no such issue, then the pattern yields a Type 2 intransitivity.

In the voting pattern above, issue 1 is tested first. It is preceded by issue 2 in rows II, III, and IV; thus, it is not the majority issue. Next issue 2 is tested. It is preceded by 1 in only two rows (I, V), and by 3 in only two rows (I, II), hence it precedes all other issues in more than half of the rows. Issue 2 is the majority issue and the pattern yields a transitive social ordering.

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